Invertibility of 2×2 Matrices

Theorem 1: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if and only if $ad - bc \neq 0$.

Proof Part 1: Show that if $ad - bc \neq 0$ then A is invertible.

$$B = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$AB = \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \cancel{\frac{1}{ad-bc}} \begin{bmatrix} \cancel{ad-bc} & \cancel{-ad+ab} \\ \cancel{cd-cd} & \cancel{ad-bc} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$BA =$ <u>exersize</u>

<u>Conclusion</u> A is invertable and

$A^{-1} = B$

Proof Part 2: Show that if $ad - bc = 0$ then A is *not* invertible.

Hint: If there exists a *nonzero* vector $\boldsymbol{x}$ in null(A), then A is not invertible.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose $ad - bc = 0$.

Case 1) $a = c = 0$

$$\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

Case 2) $a \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} \underbrace{-ba + ba}_{0} \\ \underbrace{ad - bc}_{0} \end{bmatrix} = \vec{0}$$

Case 3) $c \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} \underbrace{ad - bc}_{0} \\ \underbrace{cd - cd}_{0} \end{bmatrix} = \vec{0}.$$

Theorem 1: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Leftrightarrow$ $ab - bc \neq 0$

Definition: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The *determinant* of A is defined by $\det(A) = ad - bc$.

Corollary 1: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if and only if $\det(A) \neq 0$.

Example: Which of the following matrices (if any) are invertible? Explain.

- $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ $\det\left(\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\right) = (1)(0) - (2)(-1) = 2 \neq 0$

  A is inventable by corollary 1.

- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = (1)(4) - (3)(2) = -2 \neq 0$

  A is inventable by corollary 1.

- $\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ $\det\left(\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\right) = (1)(2) - (-2)(-1) = 0 = 0$

  A is not inventable by corollary 1.